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An Arithmetic and Its Geometry in the Higher Degrees of Laws of Form

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The enigma of George Spencer Brown's *Laws of Form* (1969) is that it claims to be presenting an arithmetic. This is an arithmetic for Boolean algebra, an underlying combinatorial structure based on distinction that is seen to underpin logical algebra. In Spencer-Brown's arithmetic there are no numerals. It is a non-numerical arithmetic, and yet it can also be used (by restricting its relations) as a form of ordinary arithmetic. Nevertheless, the way from the formal arithmetic of Spencer-Brown to the arithmetic of the natural numbers and to classical geometry is not yet fully explored. This paper finds a way to understand Spencer Brown's project in the original idea of arithmetic as practised by the ancient Pythagoreans. It shows how their emanation of dimensional magnitudes guides us to an order in the higher degrees of formal arithmetic where a geometry is found to emerge. The investigations of this geometry in the plane are very preliminary, nevertheless they suggest that there might be a way to classify geometric and algebraic numbers by degrees of infinity according to the degree of number in the formal arithmetic that is used to express them.
Keywords: laws of form; re-entry; form dynamics; arithmetic; infinity.

Preface

Fifty years ago, George Spencer Brown's *Laws of Form* (1969) introduced a new arithmetic with binary value in a unary form. This article uses interpretations of laws of form higher degree arithmetic to find an order of numbering that is applicable to mensural geometry. The higher degrees of laws of form have previously been explored by others, with much of the pioneering work on form dynamics undertaken by Louis Kauffman since a ground breaking article published with Francisco Varela 40 years ago (1980). Kauffman (1987, 2019) has offered many analogies, notably in topology and with the imaginary values in algebra. As for arithmetic qua numbering, Spencer Brown (1961, published in 1997) made an early effort to express the natural counting numbers in a way that permitted multiplication and addition, and this was subsequently progressed by Kauffman (1995, 2011). Other related developments include: an arithmetic of containment by Jeffrey James (1993) complete with geometric numbers, real and imaginary; and the *boundary mathematics* that William Bricken (2019) has used to develop principles for *iconic arithmetic*. The interpretations of laws of form in this article are mostly derived from Kauffman's explorations of both form arithmetic and form dynamics, while much of its philosophical and historical background is found in *Enthusiastic Mathematics* (Lewin, 2018).

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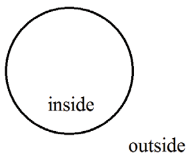
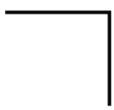
1. Introduction to *Laws of Form*

We start with a brief introduction to *Laws of Form*. Its aim is only to orientate the reader to this exposition. Those unfamiliar with *Laws of Form* may wish to first consult another more general introduction (Kauffman, 2019; Burnett-Stuart, 2013) and/or the book itself (Spencer-Brown, 1969).

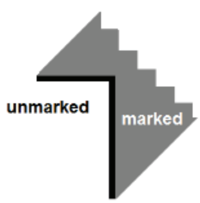


Laws of Form begins with the act of making a *distinction*. This can be imagined as a circle cutting a plane space into two sides with differential value.



In the unary notation of the *form*, the *mark* retains the topology of the distinction.

The first distinction on the page indicates the *value* of the outside as *marked*, while the inside is *unmarked*.



From here, a *calculus of indication* may proceed by either *calling* a mark again, or by *crossing* the first mark.

In the notation, calling is expressed thus: ⌉ becomes ⌉ ⌉. This is to note the repeating or copying of a distinction. When a Mark is crossed, ⌉ becomes ⌉⌉.

A crossing from one side of the distinction to the other affects a marking in the space of the first that changes the value of the expression. Cross back again and the expression returns to its original value. Thus, a double crossing of a mark effects a mark-in-a-mark-in-a-mark, which returns the expression to the value of the original mark. In the same way, the first crossing of the mark of the first distinction can now be seen as a return of the expression to the value of the unmarked page. This dynamics of crossing can be easier to understand when the form is interpreted for the logical NOT. The mark of the first distinction negates the unmarked space in its very marking. Marked is un-unmarked. Crossing this first mark is a double negation. Un-un-unmarked returns the value of the expression to that of the unmarked page.

Crossing creates a new depth of space. Cross again and another inner depth is created, and so on, with the value within the expression alternating through the depths. The two laws of the arithmetic are now apparent:

Law of Number	**Law of Order**

According to the law of number, to call or call again does not change the value of an expression. Thus, both side of the ⇔ are marked. According to the law of order, a cross changes the value. Thus, a mark-in-a-mark has the value of no mark at all.

Note how the simplest indication of the unmarked state is not a cipher, but only the unmarked page. While this can be confusing, it allows the arithmetic to express absence by absence. In the long history of mathematics, the marking of an absence with a presence was introduced whenever calculation moved from physical calculators to the page. The notational zero was a useful contrivance that these laws of form manage to avoid.

Spencer-Brown shows how these two laws of order and number underlie conventional Boolean logic and algebra. But he does this only in an appendix to *Laws of Form*. This is because the logical TRUE/FALSE binary is but one interpretation of the marked/unmarked value in a more general indicative calculus operating according to the two simple laws of calling and crossing. These laws redefine the formal sciences in a new order. This order begins with arithmetic arising from the topology of distinction that is binary in a Boolean sense. From the arithmetic an algebra develops. But this is not Boolean algebra as we know it. It is an algebra of the arithmetic with general application, not specifically logical. Thus we have the primary arithmetic, its algebra and its logic all arising from the topology of distinction. But what of that other formal science, mensural geometry?

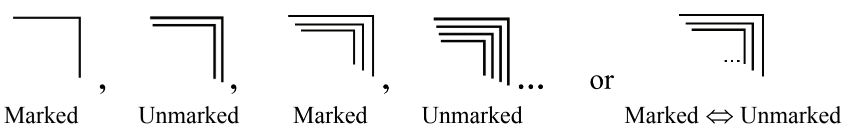
After publishing *Laws of Form*, Spencer-Brown spent much effort with applications to topology (e.g., to prove the 4-colour theorem) and number theory (e.g., to prove Riemann's hypothesis). However there is little in the way of development towards a general application to the measurement of geometric figures. In a preface published in 1979, Spencer Brown said that *Laws of Form* considers "an arithmetic whose geometry as yet has no numerical measure" (p. xi).

Conventional arithmetic moves so quickly and easily into geometry that the transition can be missed. And our language smooths the way: When squares, cubes and irrational roots come into play, we still call them *numbers*; and the continuum upon which geometric figures are represented in analytic geometry is not a continuum of measures but of numbers, indeed, of *real numbers*. This is before the algebra of higher degrees introduces numbers situated on a real-imaginary plane. Descend back down from this complex plane to the line, to the fully real number line and then to the rational number line and we have not yet left linear geometry. Indeed, we are still in a geometric interpretation of arithmetic when it is only the positive and negative integers ranging out in opposite directions from the number zero. It is only after descending again to the positive whole numbers (and their rational relations) that we have finally arrived back at the purely arithmetic. Above that, the conventional hierarchy of number provides a ready-made interpretation of arithmetic for geometry and algebra.

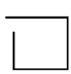
Since *Laws of Form* was published, some fascinating symmetries have been found between its arithmetic and this conventional structures of geometric numbering (Kauffman, 1987, 1995). But the question arises whether geometric analysis and geometric arithmetic might structure differently if it were developed directly from the form. And that is exactly what is attempted here. We build a geometry directly from the formal arithmetic. This is not as difficult as it might first appear because the arithmetic is found to have its own shape ripe for interpretation. But this only becomes apparent in its higher degrees.

2. Infinite Expressions

The arithmetic enters its higher degrees with the introduction of infinite expressions. The elementary number in this infinite arithmetic is the form of a distinction entering its own negative space. *Re-entry of the form* is an elementary recursion, a self-NOT-ing, as the form repeatedly crosses itself to generate this series:



The first term in this series is a single mark in the zero depth of the unmarked page. Its inside is depth 1. Next consider the second term, the two nested marks. Its inner space is two crosses from the zero depth and so depth 2. The inner space of the third term is depth 3. And so forth. Also notice that the law of crossing cancels the value of two nested marks, and so the second term returns to the unmarked state. Two nested marks inside two nested marks is also in the unmarked state, as are all even members of this series with value oscillating marked↔unmarked in the procession of elementary re-entry.

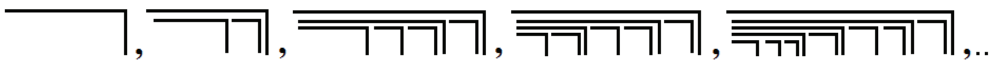


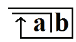
The higher degree arithmetic arrives in chapter 11 of *Laws of Form,* where Spencer- Brown introduces a modification of his mark to express re-entry.

The addition of an arrowhead emphasizes the direction of the generative process.

In a complex expression, re-entry can occur at any inner space of a distinction. One important example is:

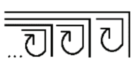
Its pattern of generation is apparent in an analysis of successive depths:

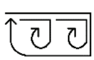


This expression is of an archetype with the algebraic form  where a and b can be any sub-expression, whether finite or infinite. This is in fact the first re-entry expression described in *Laws of Form,* introduced on pages 56 and 65 as “E1,” and we will return to it below.

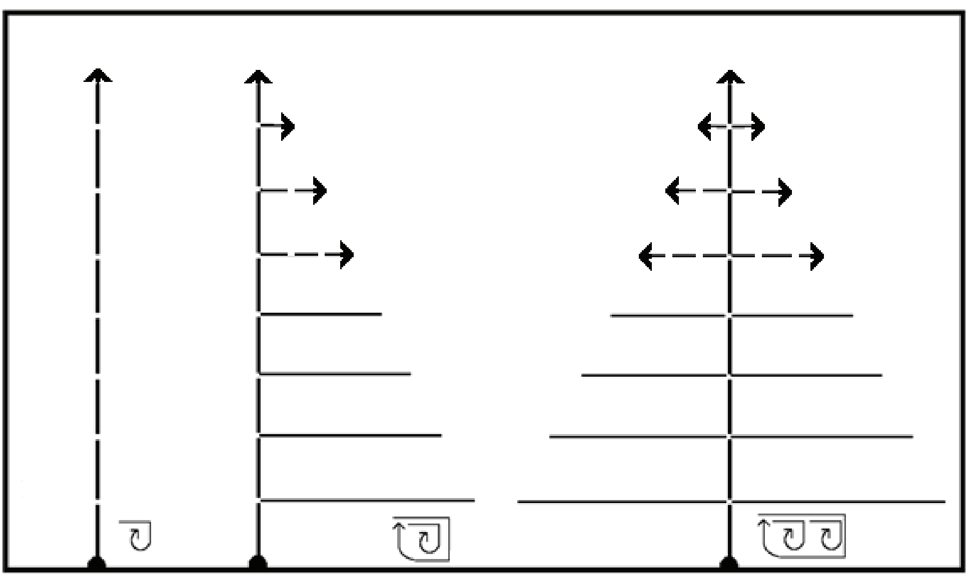
Infinite expressions within infinite expressions build the degrees. That these are infinities-of-infinities is easier to see when using a more geometric notation.

 First, consider this elementary expression in the 2nd degree.

 This generates an infinite series of primary re-entries.

 Another simple expression generates an infinite series of two primary re-entries.

The expansion of both these expressions can be visualised in *tree notation*:



First consider elementary re-entry notated on the left. Each successive period is expressed by a line segment proceeding vertically. In the middle is elementary re-entry in the 2nd degree. It has the same vertically procession (trunk), but this procession is also repeated horizontally (branches) at each of its periodic nodes. On the right there are two of these branches at each node. Consider that these trees potentially grow up forever. The two on the right grow up and out forever. As infinite series (trunk) of infinite series (branches) they are said to be in the 2nd degree of infinity. Together these three infinite arithmetic expressions are at the core of the geometry that will now be introduced.

3. An Order in the Finite Arithmetic

In laws of form, anything that is not forbidden is permitted. What is permitted must conform to the form of distinction and its laws of number and order. Further restrictions may be added to support interpretations. Interpretations may be used in particular applications. An example of an application is Boolean logic, where the marked/unmarked value is interpreted as TRUE/FALSE (see Appendix 2 of *Laws of Form*).

Another application interprets for the natural numbers series thus:

1, 2, 3, ,…

Counting is calling at the first depth. The counting numbers are expressed by the number of empty marks in a containing mark. In this interpretation, the restrictions on the laws begin with the prohibition on calling at the first depth. This serves to retain the distinction between the successive numbers. Further axioms mean that the operations of multiplication and addition are achievable without notational operators through simple processes in a restricted calculus of indication (Spencer-Brown, 1961/ 1997; Kauffman, 1995).

A very different interpretation is used for the numbering that we now introduce to support our mensural geometry. It also involves prohibitions on calling, but what makes it special is its departure entirely from the calculus of indication. Instead, the re-entry-of-the-form-into-its-own-inner-space defines the arithmetic and its geometry. At its root and in every trunk and branch is always found the tick↔tock of the marked↔unmarked alternation. The primary ordinal series meters this temporal emanation:

∞ : 1st, 2nd, 3rd, 4th...

Thus, the arithmetic arises from an emanation that is formally unending. Its finitude and the finitude of its geometry comes through delimitation: The re-entry can be stopped or truncated (delimited) whenever and however we choose. So, we now have two types of limitation. Firstly there is the distinction, our elementary difference, which is optimally expressed by a spatial limit (e.g., a circle in a plane). Then there is this arithmetic of differentiation, which is a temporal delimiting of the unending process of distinguishing. G. W. Leibniz [1646-1716] described this limited-nature-of-things, both spatial and temporal, and called it their *non plus ultra*. Our every thing is distinguished by not being more or otherwise. Consequently, and paradoxically, the exception is absolute unlimited being. Being without negation can be described *via negativa* as no-thing. It is undistinguished no-thing-ness. Every thing distinguished out of this absolute negative form carries a vestige of its origin as its not-self. This opposite-that-completes-the-form-of-things is hidden in their very nature, and so it is often lost to the formal sciences. Leibniz's attempt to keep it in view with the notational zero was not entirely successful. (Lewin, 2018, p. 283–289) We try again, this time with laws of form notation, and persist with it even when this can be cumbersome. Of course, once the formal mathematics is established, convenience will demand that our direct topological expression of the form is obscured behind notational symbols and signs, which should nevertheless continue to respect its laws.

Having said that arithmetic is delimited infinitude, for the purposes of exposition, we will still begin in the finite arithmetic with expressions that do not emphasis their truncated nature. This finite arithmetic is not more elementary, but it is likely more familiar to the reader, especially those familiar with *Laws of Form*. Only note one difference in terminology. *Laws of Form*'s chapter "Equations of the Second Degree" involves expressions that have only one depth of re-entry and so we would say they are in the 1st degree of infinity. Thus, our 1st degree equates with *Laws of Form*'s 2nd degree. As for the finite expressions filling most of the pages in Spencer-Brown's book, they will be regarded as delimited infinitudes. And so it is with such pre-cut finite expressions that we begin the interpretation that we will call Boolean arithmetic.

Table 1: Simple finite numbers (Lewin, 2018)

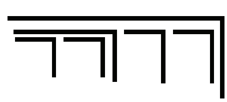
		NUMBER			
		0	1	2	3
ORDER	0				
	1st				
	2nd				
	3rd				
	...	...	...	...	...

In Table 1, consider firstly the row of 0 order simple finite numbers. The count of empty marks determines what we will call the Brownian number of the expression. Each mark is like a pebble on the sand. Now consider the 1st order. The pebbles are now in buckets. In general, what we will call the *Brownian order* of an expression corresponds to its depth, and *Brownian number* is the cardinal count of marks at the deepest depth. Thus, the bottom row gives Brownian numbers 1, 2 and 3 in the 3rd order by a count of 1, 2 & 3 marks, respectively, in the 3rd depth. This table provides only the first three simple finite numbers in the first three depths, but there is no end to number or order.

Next consider the columns where the same number is found at successive depths. Notice how the expressions in the 0 column have been drawn to appear empty at the relevant depth. For example $0^{\text{Order 3}}$ has no marks at depth three. But also notice how this expression has the same form as $1^{\text{Order 2}}$. Similarly, $0^{\text{Order 2}}$ is the same as $1^{\text{Order 1}}$, and in general $0^{\text{Order n+1}} = 1^{\text{Order n}}$. This is because in arithmetic qua arithmetic, zero is not a number but only an absence. We use the zero column here only to show the empty space in which numbering may commence with the first cross into it.

Finally, consider the value of the expressions in Table 1, that is, whether they are in the marked or unmarked state. We will notice that simple number expressions of the same order are in the same Brownian state, that is, they have the same Brownian value. Zero-order numbers are in the marked state. First order numbers are unmarked. Second-order are marked, and so forth, alternating through the orders. By the law of calling, increasing the number of empty marks in the same space does not change the value state of an expression. But complex expressions of the same order are not all in the same value state. While noting these considerations of value, and as we move to consider complex finite expressions, it is worth also noting that the value state of spaces or expressions does not play such a critical role in this interpretation as it does in, say, TRUE/FALSE logic.

So far, with these simple numbers, there is only one mark in the shallower depths. But compound numbers can have different tallies of marks at different depths.



Consider this 2nd order expression that is 2 at the 2nd depth but also 3 at the 1st depth. The count of empty marks at the deepest depth gives the *major Brownian number* of an expression. Thus, the major number of this expression is 2. In compound numbers, the tally at other depths are the *minor Brownian numbers* for that expression. The important minor numbers in this expression is 3 at the 1st depth. It also has a minor number of unity at its zero depth, which is the same as for the simple finite numbers. Indeed, for simple numbers, the minor numbers are all unity.

So far we have been counting, or tallying marks at different depths, but we can also give a tally total of all the marks in an expression. Thus, in this example the tally totals at 6. Of course, the tally for an expression is not at all unique to that expression.

In fact, the similarity of expressions in terms of their tally profile is important to this interpretation. They can be similar in their major number, in their tally total or in their entire tally profile, where their major and minor numbers are exactly the same. Where these *tally similarities* are especially important is where conventional arithmetic is found to interpret them as equivalences. Some examples of this can be found in processes analogous to the simple operations of ordinary arithmetic.

4. Analogues of Addition and Multiplication

Addition is performed by placing expressions beside each other. For example:

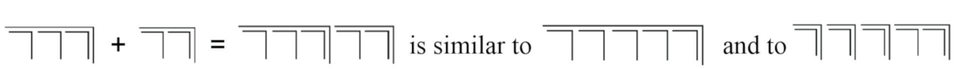
The simple operation of linking (or concatenating) expressions is familiar to symbolic logic but presents a risk of confusion due to implicit operators in mathematical algebra. In conventional algebra $ab = a \times b$, whereas in this arithmetic $ab = a + b$. Now consider addition of the same simple numbers in the 1st order:

Notice that the addition of major numbers works to give $3 + 2 = 5$, but the minor unity has also been added and so our sum has tally identity with these other expressions:

These expressions all equate if interpreted simply thus:

$$3^{\text{Order 1}} + 2^{\text{Order 1}} = 4^{\text{Order 1}} + 1^{\text{Order 1}} = 5^{\text{Order 1}} + 0^{\text{Order 1}}.$$

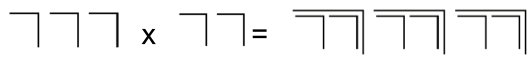
Note that $0^{\text{Order 1}}$ is more correctly interpreted as $1^{\text{Order 0}}$. It is only interpreted as an *absence* at the 1st order to support the similarity. For our purposes we might decide that empty buckets need not be counted. Indeed, if we don't care how many buckets for our pebbles, then counting pebbles is counting the major number, and so we can add other similarities:



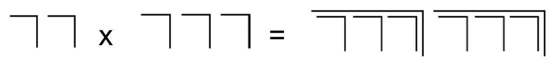
In each case the tally at the 1st depth is 5, while the tally at the zero depth varies.

With these analogues of conventional addition, we have found an accord with Boolean arithmetic. But we have also found that the ordinal aspect of Boolean arithmetic presents complications that need to be addressed or otherwise consistently ignored. The complications increase with other operations, including multiplication. Indeed, there is no direct equivalent to the operation of multiplication such that any expression can be multiplied by any other in a consistent way. However, it is useful to consider some processes that have characteristics of conventional multiplication. For now, let’s consider two of them.

The first process similar to multiplication is only successive addition. Consider that ⏋⏋ + ⏋⏋ = ⏋⏋⏋⏋, as “two twos are four,” is like 2 x 2. Likewise, ⏋⏋ + ⏋⏋ + ⏋⏋ = ⏋⏋⏋⏋⏋⏋ is “three twos,” which is like 3 x 2. Another process similar to multiplication is a particular crossing instruction.[2] This is the instruction to cross the second number into each unit of the first, thereby giving the product at a new depth. For example, 3 x 2 would be:



Now consider it the other way around, 2 x 3:



In both cases the tally at the 1st depth delivers the product 6. Yet the two expressions are clearly different (even if they are both in the unmarked state). It is only with regard to the major number that we can say 3 x 2 = 2 x 3. But also note that these two expressions preserve the difference between, say, placing 2 dumplings on 3 plates and placing 3 dumplings on 2 plates. Each operation requires 6 dumplings but they are distributed differently. It might be that when we conventionally say 3 x 2 = 2 x 3, we are conveniently ignoring an aspect of the formal arithmetic, which we can also choose to do when calculating directly in the form.

As we will see later, that is precisely how it works: such *logistical calculations* work by abstracting equivalence from the primary arithmetic. These basic applications of tally analysis are only introduced here to show how conventional arithmetic is a calculus of convenience abstracting some aspects of the formal relations of things while tending to obscure others. This should become more evident in our discussion of *logistic* below, but we are now ready to consider this interpretation in its higher degrees.

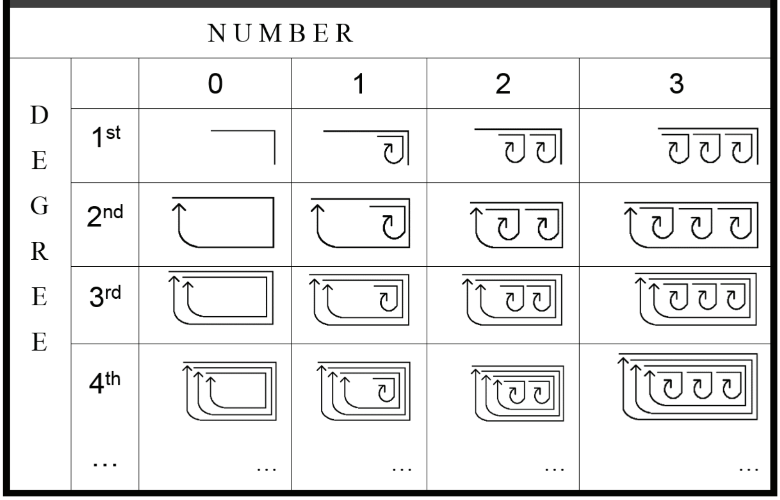
2. See Kauffman, 1995, p. 13–14, for a consistent way to permit multiplication through restrictions on the *calculus of indication* that has similarities to the process discussed here.

5. An Order in the Higher Degrees

The higher degree numbering is found similar to the finite numbering by noticing that in place of "Order" is "Degree." In Table 2, consider the first row, or 1st degree. The count of elementary re-entries determines the Brownian number of the expression. Now consider the higher degrees. The degree of the expression corresponds to the depth of re-entries. Thus, number 1 in the 3rd degree has one elementary re-entry within a re-entry within a re-entry. Number 2 in 3rd degree has two elementary re-entries at the same depth. And so forth.

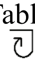
Next, consider the columns where the same number is found at successive depths. Notice how the expressions in the 0 column have been drawn to appear empty at the relevant depth. For example $0^{\text{Degree } 4}$ has no marks re-entering depth four. But also notice how this expression has the same form as $1^{\text{Degree } 3}$. Similarly, $0^{\text{Degree } 3}$ is the same as $1^{\text{Degree } 2}$, and in general $0^{\text{Degree } n+1} = 1^{\text{Degree } n}$. Again, as with finite Order, this zero column is mainly to show that zero is not a number but only the place where numbering might begin. But its presence also helps with understanding the symmetry of the entire hierarchy, and especially where this identity of 1 and 0 across the degrees is extended to the orders of the finite arithmetic.

Table 2: Simple infinite numbers (Lewin, 2018)



In the first place, this link with finite numbering comes through the elementary mark, ⏋. It is 0 in the first degree of infinity, but it is also $1^{\text{Depth } 0}$ and $0^{\text{Depth } 1}$ (see Table 1). In the second place, the link comes through elementary re-entry, . To see how this works, consider again our foundational analogy:

∞ : 1^{st}, 2^{nd}, 3^{rd}, 4^{th}...

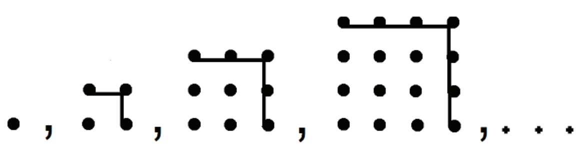
This series is found in column 1 of Table 1 as unity in the successive orders/depths. Notice that its infinite expression is . Next, notice also how this infinite one is number 1 in the 1st degree of infinity. Finally, notice once again that this is also the empty place of numbering in the 2nd degree. Thus:

$$\infty^{\text{Degree 0}} = 1^{\text{Degree 1}} = 0^{\text{Degree 2}}.$$

The elegant symmetry of this ordinal hierarchy suggests that it could hardly be new to science. Indeed, it is not. It might even be as old as mathematics itself in as much as it is implicit in a hierarchy of dimensional magnitudes used by the ancient Greek *mathematici*, the Pythagoreans. Consideration of their hierarchy introduces the shape of our arithmetic.

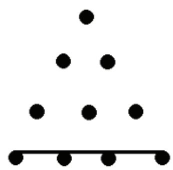
6. The Pythagorean Gnomic Expansions

Pythagorean mathematical emanationism teaches that all difference is generated out of a principle-origin. This original unity is an unlimited-limit that generates by self-limiting. Their ordinal counting expressed this emanation-by-limitation where the alternation between odd and even numbers was seen as an alternation between the *excessive* and *just* values, one exceeding the value of the original limit and the other returning just with it (Lewin, 2018, pp. 177–180). Little is known of Pythagorean mathematical philosophy before it was revolutionised by Plato [d. 347 BC], but we do know of their early interest in the generation of figured numbers by *gnomic expansions* as directly expressed in non-numerical notations (Heath, 1921, vol. I, pp. 76–82).



Consider firstly, the expansion generating the square numbers. *Gnomon* has the general meaning of rule, and it was what the builder called his set-square. We can see how just such a set-square might have been used to direct the placing of each new addition of pebbles or dots, thereby generating each successive square number from the previous number, and all from the original unity. Thus, gnomons of 3, 5, 7… generate each new square number, 4, 9, 16….

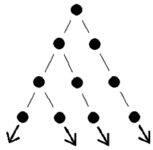
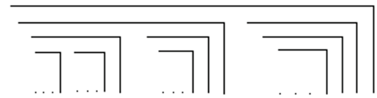
Rules with increasing number of sides were used to generate a set of figured number series, which we will come to shortly. But let's first consider the gnomic generation in the plane that was considered most elementary. This was when a simple straight rule was used to produces the triangular numbers, the third generation of which was found to have special qualities.



Some of the special qualities of this sacred *Tetraktys* will soon become apparent, only now notice that its straight ruling gnomon corresponds to the most ancient known scientific use of that term. According to legend, the vertical rod of the sundial—which is still called gnomon today—was the first scientific instrument that the first Greek student of science brought from Egypt even before Pythagoras. And so we might further imagine Pythagoras taking up this rod and some pebbles to then rule up in the sand the first among the gnomic expansions of his arithmetic. It turns out that this straight rule generator also has a special place in our arithmetic.

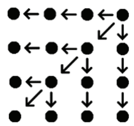
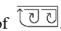
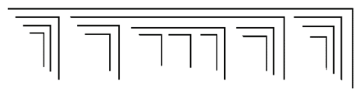
That the Pythagoreans used gnomic expansions to express a self-generation, we know. But we don't know exactly how they imagined this generation to proceed. We can only speculate that it was the simplest procession of all, which also corresponds to the form of our number 1 in 2nd degree.

Table 3: Triangular generation (Lewin, 2018)

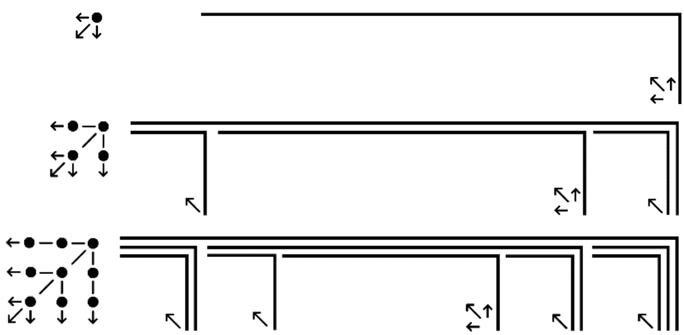
Pythagorean with speculated paths	Hybrid notation	Boolean Arithmetic as the expansion of .
		

In Table 3, see firstly the speculated path of triangular generation into the *Tetraktys* and beyond. Next, see this generation expressed as containment, where each unit contains the units that it generates. Finally, the form of generation is expressed in laws of form notation where the progression of dots have become nested marks.

In the same way, square generation is found to be in the form of our number 2 in the 2nd degree. In Table 4, the gnomic expansion of the square numbers is flipped over to show more clearly the correspondence with the generation in our notation.

Table 4: Square number generation (Lewin, 2018)	
Pythagorean flipped with speculated path 	…in Boolean Arithmetic, the expansion of . 

Here are the first three periods of the generation in both notations:



Notice how the square number at each period corresponds to our tally total, while the gnomon corresponds to our major number. For example, in the 3rd period, the square number 9 is our tally total while its gnomon 5 is our major number.

If the analogy is taken the other way around, then the figured number generation reveals the inherent shape of our arithmetic. This is especially so when we next consider how the Pythagoreans presented these emanations in a hierarchy according to the generation of the physical dimensions.

7. Pythagorean Hierarchy of Dimensional Magnitudes

For the Pythagoreans, the geometric dimensions generate from an original dimensionless point by motion in three degrees. When the point moves it generates the line. When the line moves it generates the plane. When the plane moves it generates the solid. In this generation, the plane is not part of the solid but only its origin. Nor is the line part of the plane but only its starting place. The point has no dimensionality, but is the origin of all. This hierarchy of dimensions is geometrically expressed in their hierarchy of figured numbers summarised in Table 5. The best surviving account of this geometric hierarchy is in an introduction to arithmetic by Nicomachus of

Gerasa [60–120 AD] where he makes a special point of saying that the hierarchy already exists in the arithmetic itself, that is, prior to its application to geometry (Nicomachus, 1926, pp. 237–238).

Table 5: The hierarchy of figured numbers according to Nicomachus (Nicomachus of Gerasa, 1926, Bk II, Ch VI-XIV) (Lewin, 2018)	
0-Dimension	*The point* corresponding to the original One or "Monad"
1-Dimension	*The line* corresponding to the elementary just/excessive alternation i.e., [1], 2, 3, 4, 5…
2-Dimensions	*The triangular* number series is elementary. From it are derived the square number series, the pentagonal numbers, the hexagonals, the heptagonals and so forth.
3-Dimensions	*The triangular pyramid* number series is elementary. From it are derived the square pyramid numbers, the pentagonal pyramids, the hexagonal pyramids, the heptagonal pyramids and so forth.

The analogy with our arithmetic begins in the first dimension with the linear numbers compared to the elementary mark re-entering its own negative space and generating an ordinal series with alternating value. The analogy continues where each of their plane number series is orderly aligned with our 2nd degree numbers as summaries in Table 6.

Table 6: The generation of plane figured numbers (Lewin, 2018)

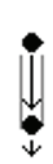
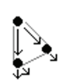
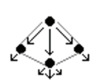
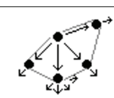
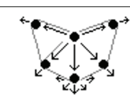
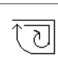
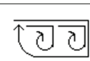
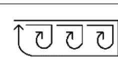
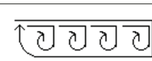
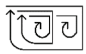
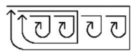
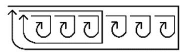
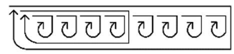
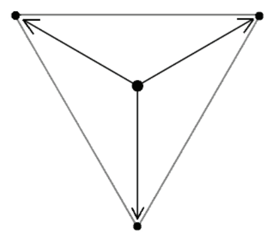
	[Lines]	Triangles	Squares	Pentagonals	Hexagonals
The primary generation					
Expressed as Re-Entries					
Number series	1,2,3,4,5…	1,3,6,10,15…	1,4,9,16,25…	1,5,12,22,35…	1,6,15,28,45…
Gnomon series	1,1,1,1…	2,3,4,5…	3,5,7,9…	4,7,10,13…	5,9,13,17…
Gnomic interval	0	1	2	3	4

Table 6 introduces derivative analysis to support the comparison. The 1st derivative, the gnomon series (or major number series), we already know from the previous examples as the number of pebbles/dots/marks added with each period of the expansion to give the triangular and square number series. The 2nd derivative is the successive intervals of the gnomon series or the *gnomic interval*. Notice how this 2nd derivative expresses the 2nd degree number.

Table 7: The generation of the solid figured numbers (Lewin, 2018)

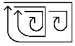
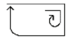
	[Triangular]	Triangular Pyramid	Square Pyramid	Pentagonal Pyramid	Hexagonal Pyramid
					
Number series	1,3,6,10,15,…	1,4,10,20,35,…	1,5,14,30,55,…	1,6,18,40,75,…	1,7,22,50,95,…
Gnomon series	2,3,4,5…	3,6,10,15…	4,9,16,25…	5,12,22,35…	6,15,28,45…
2nd derivative	1,1,1…	3,4,5…	5,7,9…	7,10,13…	9,13,17…
3rd derivative	0	1	2	3	4



For the Pythagoreans, the first solid number is a triangular emanation from a point which generates a tetrahedron or triangular pyramid. Out of this proceeds larger and larger pyramids with bases of the successive triangular numbers. Next comes the series of square-base pyramid numbers. After these 4-sided pyramids come the 5-sided pyramids, 6-sided pyramids, and so forth. Table 7 shows how these series correspond to our numbers at 3rd degree by comparing the tally of re-entries into the 3rd depth with the 3rd derivative.

But also notice in Table 7 how the elementary re-entries into the 2nd depth mirrors those into the 3rd. For example, the generator of the hexagonal pyramid numbers has four elementary re-entries into the 2nd depth as well as into the 3rd. This can be understood in several ways. For now, consider it as a higher form of re-entry where the higher degree number is the same lower degree number containing a copy of itself in its shallowest space. For example, consider how unity in 2nd and 3rd degrees can be derived from 1st degree unity:

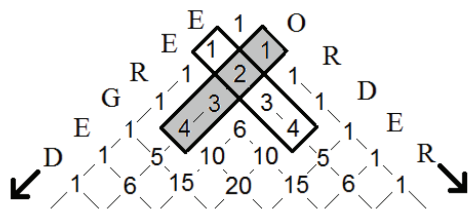
The plane  is the line  containing a copy of itself in its inner re-entry space.

The solid,  is the plane  containing a copy of itself in its shallowest re-entry depth alongside its re-entering line.

This containment expresses a simple and familiar symmetry that the Pythagoreans sometimes saw as a stacking. The triangular pyramid numbers are stacks of successive triangular numbers. Stacking the squares builds the square pyramids. And so forth. They also saw this symmetry in the degrees of cumulative addition. We can see this in the derivative analysis by following each Brownian number up the hierarchy of degrees and noticing how one series is the 1st derivative of the next. Let's see how this works for unity corresponding as it does with the Pythagorean's elementary generator in each dimension:

1	The number of the origin
1 , 1, 1, 1, 1…	Each gnomon of the linear series
1 , 2, 3, 4, 5…	The tally in the linear generations, or the gnomons of the triangular series
1 , 3, 6, 10, 15…	The tally in the triangular generations, or the gnomons of the tri' pyramid series
1 , 4, 10, 20, 35...	The tally of the tri' pyramid generations, or the gnomons of the 4^{th} degree series

The pattern could be continued into even higher degrees, but this is sufficient to notice that it is of the form of Pascal's triangle, with ordinal depth on one axis and degree on the other.



In this version of Pascal's triangle, as the series in successive degrees build by cumulative addition, so too the series in successive depths build across the degrees, thus giving a table symmetrical about a central vertical axis. In this symmetry, we can see how the *Tetraktys* of the 2^{nd} degree (boxed) provides a neat summary of the first three degrees of the arithmetic through its similarity with the series at the 1^{st} ordinal depth (boxed and shaded). That is, the 1, 2, 3 & 4 across the first order are the first-born numbers for the unitary values in the three degrees of arithmetic, starting with the 1 for the point, through the first linear number (2), plane number (3) and solid number (4), thus completing the three physical dimensions.

8. Similarities Through Tally Analysis

The divergences of the Pythagorean hierarchy from our simple numbers hierarchy is not problematic. Rather, it points to some important symmetries that emerge in the 3^{rd} degree. These symmetries are found via analysis much like the tally analysis of the finite arithmetic in section 4 above.

Table 8: Number 2 in 3rd Degree (Lewin, 2018)

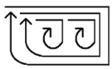
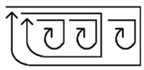
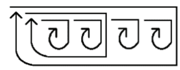
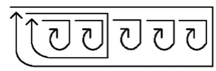
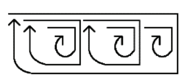
 Simple $2^{\text{Degree 3}}$		 Square Pyramid	
1, 3, 8, 18, 35, 61… 2, 5, 10, 17, 26… 3, 5, 7, 9… **2, 2, 2…**	1, 4, 11, 24, 45, 76… 3, 7, 13, 21, 31… 4, 6, 8, 10… **2, 2, 2…**	1, 5, 14, 30, 55, 91… 4, 9, 16, 25, 36… 5, 7, 9, 11… **2, 2, 2…**	1, 6, 17, 36, 65, 106… 5, 11, 19, 29, 41… 6, 8, 10, 12… **2, 2, 2…**

Table 8 gives variations of number 2 in 3rd degree, It includes our simple 2 (see Table 2) and the 2nd of the Pythagorean solid numbers, the square pyramid series (see Table 7). By counting the number of re-entries into the 2nd depth, notice that these expressions are arrange in order of minor numbers 1, 2, 3 & 4, respectively, in their 2nd degree. Next, consider the analysis below each expression and notice the influence of these minor numbers only until the 3rd derivative. If only major number were of concern, then only the 3rd derivative would matter.

These versions of $2^{\text{Degree 3}}$ are all similar in terms of their major number, but what about similarity across the entire depth profile? Take as an example the square pyramid generator and find another expression that has the same tally of re-entries into each depth. Here is one:



Just like the Pythagorean square pyramid generator, this expression has 1 re-entry into the 1st depth, 3 into the 2nd depth and 2 into the 3rd. This means that it will also generate the square pyramid number series, which is to say it will have exactly the same tally (and derivative) analysis. In general, expressions with the same re-entry tally depth profile have the same tally analysis. While this rule seems to hold more generally, it is only proposed here for the type of higher degree expressions currently under consideration.

Rule of Tally Similarity
For higher degree expressions consisting only of marks re-entering their shallowest inner depth, those with the same number of re-entries at each depth with have the same tally analysis.

This rule of tally similarity will be especially useful when comparing numbers in 3rd and higher degrees, where it can also be used in what we might call *similarity equations*. Let's first see how this works with an equation of two plane number generators:

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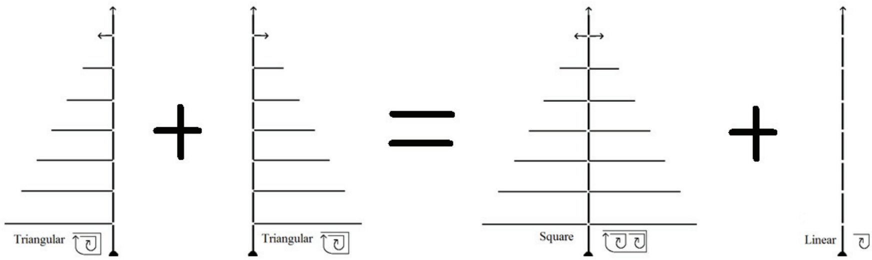


Tree notations is used here to show the sum of two triangular number series. The two triangular generators are drawn in mirror image to better show how the addition works.

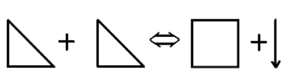
On the left side, each triangular number series have the tally profiles of 1 re-entry into depth one and 1 into depth two, thus the total is 2 at each depth. The square series on the right side has 2 re-entries into depth two but only 1 re-entry into depth one. If one linear series is added to the square series then the equation balances to give the total of 2 re-entries at each depth on each side. Figuratively this equation can be understood by first noticing how each of the triangular series has a trunk and so there are two trunks on the left side. Thus, if the two triangular trees are joined to make a single square tree, then there is one trunk left over. Numerically the equation is:

$$1^{\text{Degree 2}} + 1^{\text{Degree 2}} = 2^{\text{Degree 2}} + 0^{\text{Degree 2}}.$$

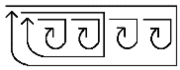
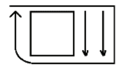
The zero is of course better expressed as $1^{\text{Degree 1}}$. Again, we only call it zero here to make the equation look balanced. Any zero remainder needs to be included when concerned with full tally similarity. It is not important, nor included when concerned only with the major number of the expressions. (Compare this with the similarity in Table 8 and also with the analogues of addition and multiplication discussion above in section 4.)

Zero remainders also appear when adding other plane figured numbers. If two square series are added then there is one zero remainder. A square joined to a triangular series also has one of these spare trunks. Add any two plane series leaves one trunk. If three plane series are added, then there are two joins and so two spare trunks. Add four leaves three. The number of zero remainders is always one less.

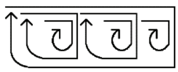
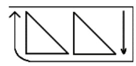
Before considering more complex examples of such tally similarity, it helps our analysis to simplify the expression of this similarity by using geometric symbols for the linear, triangular and square series. The equation given above in tree notation is now given here using these geometric symbols:



This is read thus: two triangle series are similar to a square series plus a line. Now, let’s return to 3rd degree expressions by considering the generator of the square pyramids:

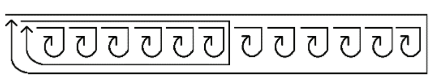
 which is .

According to our tally similarity rule, we have already found another generator of the square pyramid series:

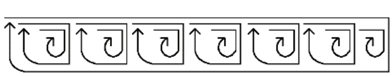
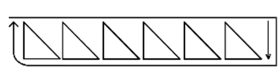
 which is .

The derivation of this alternative square pyramid generator from the original Pythagorean generator can be understood figuratively: the separating out of the two triangles has used up one of the spare trunks. In fact, the generator of all the pyramid numbers can be substituted in this way so that they are expressed as the re-entry of so many ’s with one ↓ remainder. This is because , or , has the major and minor number of unity, while all the pyramid number series have a 2nd degree number one more than their major 3rd degree number. For example the square pyramid, , is 2 in 3rd degree but 2+1 = 3 in 2nd degree.

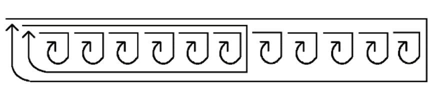
Consider as another example, the octagonal pyramid series. Here is our generator of these 8-sided pyramid numbers:



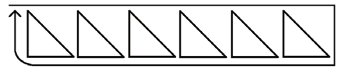
In our arithmetic this is 6 at 3rd degree but with a minor number in 2nd degree of 6+1 = 7. Thus, if we were to find its similarity with an expression re-entering a set of ’s, then there will be one ↓ remaining:

 or .

Now consider a close relative of these expressions, one that tallies 6 in both 3rd and 2nd degree:

.

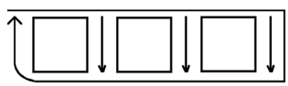
Convert to re-entering  and this time there is no remainder:

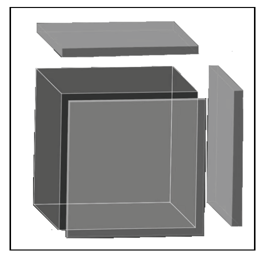


It turns out that this expression generates the cubic numbers [1^3], 2^3, 3^3, 4^3, 5^3….

The cubic series is an outlier in the Pythagorean triangular pyramid hierarchy, as it is in our simple number hierarchy, and yet here we find a shapely symmetry. Imagine a trunk with infinite triangular leaves emanating hexagonally out of every node. More arithmetically speaking, the cubic number series is generated by the re-entry of six unities in the 2nd degree.

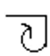
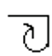
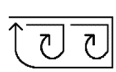
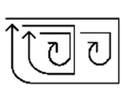
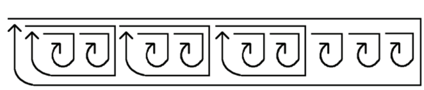
More geometrically speaking, this generation should be expressed as the layering of square surfaces on three sides of the successive cubes. And indeed, if we join together 3 pairs of the 6 triangles by our similarity rule, this gives the re-entry of 3 squares with 3 linear remainders:





It is thus in the form of three conjoined square pyramid series that we arrive at what will be the elementary 3rd degree expression in our geometry of lines, squares and cubes. This geometric hierarchy is the third hierarchy found so far in the arithmetic. The three hierarchies are summarised in Table 9. First and foremost was our simple numbers hierarchy (Table 2). Next came its variant, the Pythagorean triangular hierarchy (Tables 5–7). And now we have introduced the hierarchy that will be used in application to the geometry of square and cubic space.

Table 9: Unity in the three hierarchies

Simple Numbers	Pythagorean	Geometry
 line	 line	 line
 triangle	 triangle	 square
 simple 1 Degree 3	 tri' pyramid	 cubic

But before we move to interpret the arithmetic for geometry, it will help to take stock and reflect on our progress so far, and to reflect in particular on how our conceptions of arithmetic and its relationship to geometry compares with conventional constructions of mathematics, both modern and ancient.

9. Returning to the Ancient Conception of Arithmetic

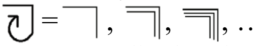
One of the great difficulties when coming to laws of form is in gauging exactly what it is that Spencer Brown is trying to show. Soon enough there is the realisation that his book presents something very elementary to the formal sciences. And the link to logic is evident if only through its appendices. But what of the link to arithmetic? At first sight its lack of numerals, its apparent resistance to the normal methods of calculation and its single operator (which is also its operand) can seem alien to modern arithmetic.

But it is much more familiar to ancient arithmetic. This is why it can be helpful to return to where the science called arithmetic originated—with the Pythagoreans. The ancients provide an environment where this new arithmetic is much more at home. As we saw above in section 6, Pythagorean arithmetic has many similarities to the higher degree arithmetic of laws of form with its emanation-by-limitation, its essential binary value and its elementary form revealed through non-numerical (pebble/dot) notation. To extend the similarity, we must understand a most fundamental way that the ancient science was different from what we moderns think of as arithmetic. What we usually call arithmetic, they saw as a derivative practical art called *logistic*. In coming to understand the relationship between logistic and arithmetic, we can see more clearly how laws of form might relate to our ordinary arithmetic, and also to ordinary square-cubic geometry.

10. Logistic

The Pythagorean science of arithmetic investigates numbers-in-themselves and how they emanate from the original unity. Logistic is the application of arithmetic to the counting of things and calculations in relation to that counting. It is the accounting of cattle, apples, bowls, armies and so forth through one-to-one correspondence (Heath, 1921, vol. I, p. 13–16). In other words, logistic analyses things already distinguished, while arithmetic investigates the generation of distinctions. The analysis can even be of arithmetic generation. That is exactly what we were doing with tallies and derivatives in sections 7–9 above: It was through logistical analysis that similarities between arithmetic expressions were found.

This application of logistic in the analysis of arithmetic generation might suggest that it is more elementary. On the contrary, the Pythagoreans and Plato left no doubt that logistic is the derivative art.[3] For us, it is easy to show how simple re-entry processes underlie logistical operations. Let's consider only basic cardinal counting.

Counting is always essentially ordinal. To count a set of objects they must be placed in an order. The 2nd must be taken after the 1st to get 2, and the 3rd after the 2nd to get 3. And so forth. Each count must be temporally inside the previous ones. Our calling is not counting. I could have called every bird I saw today (i.e., notice each as a bird) without knowing how many I saw. I could try remembering each one and then count them in my head. Or, if I now know a count is required, tomorrow I could count them as I go through the day. Imagine counting restless cattle in a pen. The count would be difficult without somehow separating off those already counted. This is why elementary counting is found to be based in our elementary ordinal series:


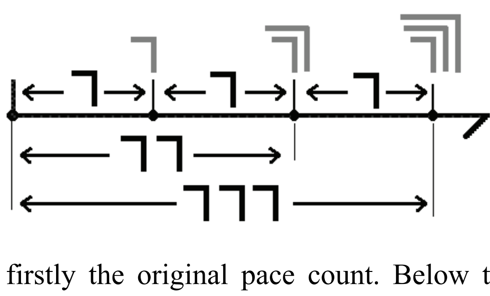
Our ordinal series can be interpreted as cardinal numbers very easily. One way to physically call cows would be by dropping a pebble in a cup as every one passes orderly through a gate. The notational equivalent of this calling-by-order would be to score a surface. Either way, if the ordinal count were not remembered then the cardinal total can only be found by another ordering.

This logistical interpretation of ordinal counting is similar to our tally analysis, where the 3rd has a tally of 3:

 translates logistically to  tally.

From the 3rd is abstracted its cardinal number of marks which is 3. The suffix *tally* will now be used to indicate that such a finite zero-depth expression is a tally total of marks abstracted from an ordinal number.

Now, instead of counting cows (or other similar pre-distinguished objects), consider counting steps. If you count your paces while walking in line then that is a fine analogy for metering linear space. Call each pace one mark's length and our mensural geometry can begin.



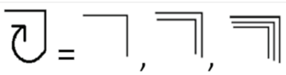
This diagram shows firstly the original pace count. Below that are the abstracted lengths of each pace given as zero-depth marks. Finally there are the lengths as tallied measures. Such linear metering could also involve going back to the start and pacing

3. (Heath, 1921, vol. I, p. 13–16) The strict definition of arithmetic as distinct from logistic and geometry remained strong into the 3rd century AD, which was when Diophantus introduced algebra into the Hellenistic tradition and called it arithmetic.

out in the opposite direction. That count could be distinguished by using negative numbers and by separating these from the positives with the starting place now "numbered" zero. Along these lines a Cartesian space can be generated for mapping discrete functional relations. None of this will be developed here in what is the briefest of introductions. Instead, we now move to introduce a new algebra designed to express such functional relations in this geometry.

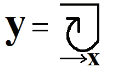
11. Algebra and the Functional Analysis of Periods

Consider how algebra might be developed for our mensuration to retain its essential ordinality. If only by analogy, order always remains inherently temporal. Discrete functions of time can be express in the form "y is a function of x" where x is the period of elementary re-entry:


..., which interprets as the periods 1st, 2nd, 3rd....

Thus for example, if x = 2 then the emanation has been delimited to its 2nd period. To see how this works, let's start at the beginning with the ratio of equality, 1:1.

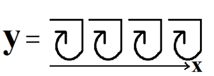
Conventionally equality is expressed in the equation, y = x. Converted to a re-entry expression delimited to the xth period, this can be expressed:



This is to say that the y count equals the count of elementary re-entry to the xth period. Thus for example, if x = 3rd, y = 3:

This equation says that y equals the tally of elementary re-entry delimited to the 3rd period, which counts to 3. Here *tally* is used as a prefix to indicate that a tally is to be abstracted from the ordinal expression. *Tally* as a suffix again indicates that this abstraction has already been made.

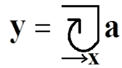
Next consider a slightly more complicated equation, y = 4x:



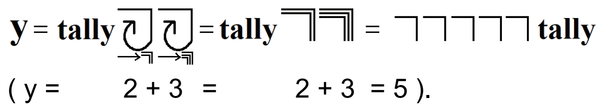
This is to say that the y count equals the tally of four elementary re-entries to the xth period. Thus, if x = 2 then:

(y = 4 x 2 = 2 +2 +2 +2 = 8).

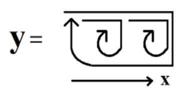
In this way, we can also translate the 1st degree algebraic archetype, $y = x + a$ thus:



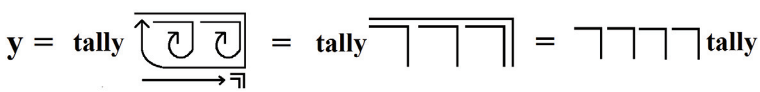
If $x = 2$ and $a = 3$:



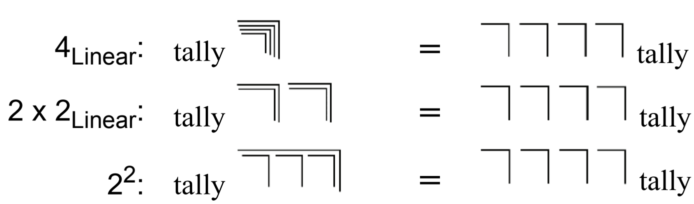
When the geometry moves to the 2nd degree, we shift away from the hierarchy of the simple arithmetic (see above, Table 9). In the simple arithmetic, unity in the 2nd Degree is the infinite . But for our square-cubic geometry, 2nd degree unity is the unit square measured as the first in the series of the square generator, . Thus, $y = x^2$ is expressed as:



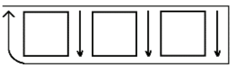
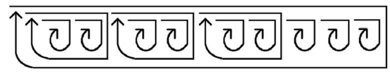
This is saying that the y count equals the tally of the square generator delimited to the x^{th} period. After the empty mark for the unit square there comes the first properly square number , which is the 2nd period of generation, where $x = 2 =$.
This is expressed:



Notice how in this arithmetic, 2^2 is not identical to the linear 4, nor to the double of 2. These are only similar by abstraction of their tallies:

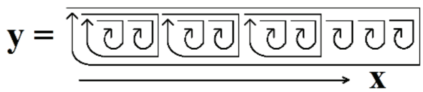


For geometry in the 3^{rd} degree, remember our cubic generator given above, in section 8:

 which translates to .

This is the expression required for cubic equations.

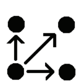
Here is $y = x^3$:



Third degree equations can then be expressed in a similar way to 2^{nd} and 1^{st} degree equations. And generators for 4^{th} and higher degree terms can also be found (although they are rather lengthy in their notational expression!) For now, equations in these higher degrees must be left for the reader to explore because there is a more pressing matter to address, as our geometry remains deficient in one elementary respect.

So far we can only express relations between measures in discrete whole number terms. While that is limiting for geometry in the 1^{st} degree, it is entirely debilitating for geometry of higher degrees where the roots of equations and diagonal lengths are mostly only expressible as fractional ratios, whether finite or infinite. Expressions for such terms can be found through an interpretation called *fractional analysis.*

12. Fractional Analysis for Finite Fractions



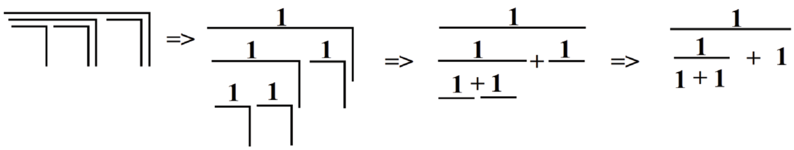
Consider again the generation of the first square number, where the 3_{gnomon} arises from the original 1. So far this has been seen as unity generating three offspring, or as a branching by trifurcation. But the trifurcation could also be interpreted as a division-into-three, where the ratio 1:3 in  is interpreted as the fraction one-over-three, or $^1/_3$. Likewise, the first triangular number  can be taken as one-over-two or $^1/_2$. And so forth.

This interpretation which we call fractional analysis can also be considered in terms of the abstracted tallies given above in section 11. It is as though you take any tally and place a mark over it to invert its rational value. In the special case of a mark over a single mark, or , this interprets as $^1/_1 = 1$. A mark over nothing is just a mark, or , which reads simple as "1 over nothing" or 1. Notice here that such a reading may appear in conflict with conventional readings of division by zero. However, it is consistent with our arithmetic where the absence of our operator-cum-operand is an absence of operation indicated by absence. It is not that $1 \div 0$ cannot be done, rather it is that it is not done. With this clarification in mind, the guiding rule of this interpretations finds simple expression:

The Rule for Fractional Analysis
$\overline{X\rceil} = \frac{1}{X}$

Using this rule, consider how more complex expressions are read as continued fractions. This is how non-unit fractions can be constructed. For example, consider $^2/_3$. It is the reciprocal of $^3/_2$, which is $^1/_2 + 1$, which in this interpretation is . Placing a mark over this expression for $^3/_2$ inverts it back to $^2/_3$, which is .

Let's come at that from the other direction. The translation of this expression for $^2/_3$ comes through its reading as the continued fraction 1/((1/(1+1) + 1). This reading can be visualised thus:



From there we can see that 1/((1/(1+1)) + 1) = 1/((1/2)+1) = 1/(3/2) = 2/3.

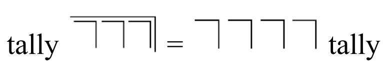
Table 10 provides samples of simple ratios expressed in this interpretation.

Table 10: Fractional Analysis interpretation for finite fractions							
$^1/_1$	$^1/_2$	$^1/_3$	$^1/_4$	$^3/_4$	$^1/_5$	$^2/_5$	$^3/_5$

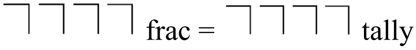
Next, consider how such fractions can be integrated into algebraic equations. Firstly, remember how fractional analysis translates as 1, and so as 2, and so forth. This is the same as for tally abstractions. Thus, if the suffix *frac* is used to denote the fractional interpretation then for example:

Both interpretations give the same result, the cardinal 3.

Consider as another example, 2^2 expressed as:



By tally abstraction, 2^2 is 4, but the tally term for 4 is also 4 by fractional analysis. That is to say:

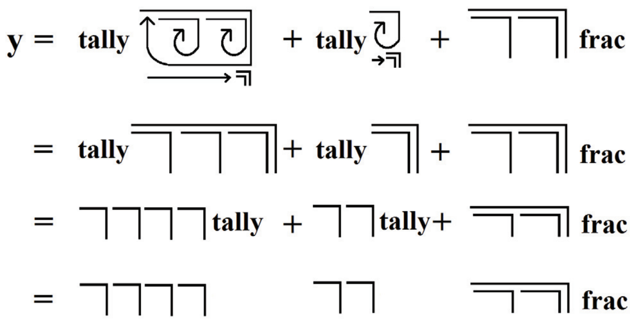


This equality is convenient in the expression of functional equations because it means that the tally expression can be reduced to, or merged with, the fractional expression.

Consider for example:

$$y = 2^2 + 2 + {}^1/_2 = 6.5$$

This can be expressed as follows:

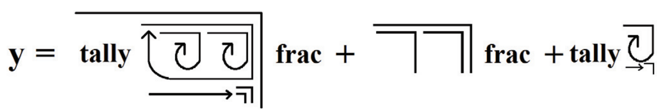


In the last line notice how the abstracted tallies, 4 and 2, have been reduced to a concatenated fractional interpretation without changing the value.

For a more complicated example consider:

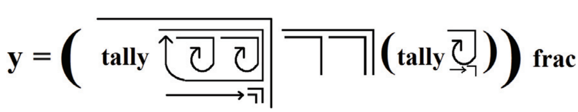
$y = x^2 + x + 1$, where x is a unit fraction.

Take the case of $x = 1/2$. In this case, $x^2 = 1^2/2^2 = 1/2^2$ and so:



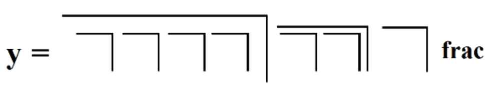
$$(\, y = \quad 1/2^2 \quad + \quad {}^1/_2 \quad + \quad 1 \,)$$

Notice that the tallies have not yet been abstracted and so the ordinal expression for 2^2 is incorporated into the first fractional expression. We can condense the statement even more by removing the plus signs and give one complete bracketed fractional expression:



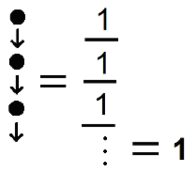
$$(\, y = \quad 1/2^2 \quad + 1/2 \quad + 1 \,)$$

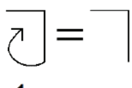
This is rather complicated and not easy to interpret for the untrained eye. However, it quickly simplifies when the tallies are abstracted:



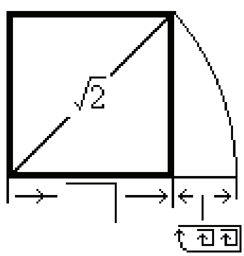
$$(\, y = \quad 1/4 \quad + 1/2 \quad + 1 \,)$$

13. Fractional Analysis for Infinite Fractions





$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}$$

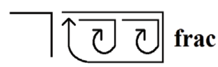


Next, consider that if this fractional analysis were applied to infinite expressions, it would present infinite continued fractions. For elementary re-entry, the continued fraction does not modify the arithmetic value, which remains at 1.

Thus, in this geometry a mark has metric value equal to the re-entered mark, that is $1^{\text{Degree } 0} = 1^{\text{Degree } 1}$. This contradictory equation is important to keep in mind for applications of this interpretation to infinite expressions.

Consider again the square number series as $2^{\text{Degree } 2}$ (see its partial expansion in Table 4), and continue the fraction it generates. The first mark generates three marks. Two of these will generate elementary re-entries, which we have just found valued at 1, giving 1 + 1 = 2. The third mark is the one that performs the re-entry of, again, 2 elementary re-entries and 1 repeat. And so forth, 1/(2 + 1/(2 + 1/…)) down the fraction.

This continued fraction turns out to be the infinite continued fraction for the diagonal of the square. That is, in this interpretation, the continued fraction for √2 is expressed as:

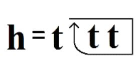


This analysis works not just for squares but for other rectangles, where their diagonals can be found in a similar way. Table 11 gives the diagonals for rectangles in ratios: 1:0, 1:½, 1:1, 1:2 & 1:3.

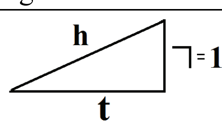
The pattern continues for unit rectangles of increasing length, and this points to a general rule:

Diagonal Rule

for finding the length of the diagonal from the ratio of the sides

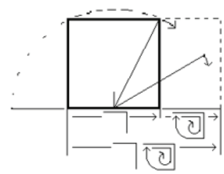
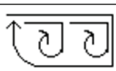
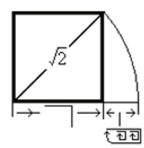
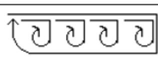
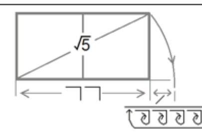
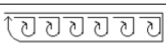
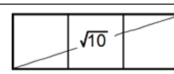


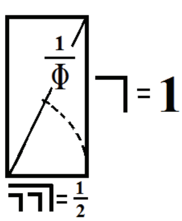
where



In this rule, h is the diagonal of the rectangle or the hypotenuse of a right-angle triangle where the sides are expressed in ratio 1:t. If we remember that in this algebra tt means t + t, then the rule will apply for all values of t, where t can be an expression for a whole number, a unit fractions, an infinite fraction, or a combination of these. This means that there is the potential to build expressions for all the geometric numbers classed as quadratic irrationals. That this in itself is nothing new or special is easier to see when the diagonal rule is translated into conventional algebra and derived from Pythagoras' theorem (see the Appendix below). Of more interest is how the diagonal rule reveals symmetries of mensural geometry that are otherwise obscured by their conventional arithmetic expression.

Table 11: 2nd Degree numbers and diagonal lengths related by Fractional Analysis (Lewin, 2018)

Number	Laws of Form Notation	Figured N° Name	Fractional Analysis giving the infinite rational value	Geometric Analogue
0		Linear 1 trunk 0 branches	$\cfrac{1}{\cfrac{1}{\cfrac{1}{\vdots}}} = 1$	
1		Triangular 1 trunk 1 branch	$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}} = \Phi$ Φ is also as $\Phi = 1+1/\Phi = 1/1/\Phi$	
2		Square 1 trunk 2 branches	$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}} = \sqrt{2}$	
4		Hexagonal 1 trunk 4 branches	$= \sqrt{5}$	
6		Octagonal 1 trunk 6 branches	$= \sqrt{10}$	



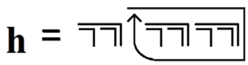
As with all interpretations, the unity affords special attention. In 1st degree, 1 generates 1. In 2nd degree, 1 generates the infinite component of the Golden Ratio, that is $\Phi - 1$ or $^1/\Phi$. Geometrically, the diagonal of the 1:½ rectangle measures ½ + $^1/\Phi$. This rectangle is proportionally half the size of the 2:1 rectangle with diagonal √5, which goes to show how the Golden Ratio and √5 are closely related. If we now consider the cases of t = ½ and t = 2, then this relationship can be seen in our arithmetic.

In conventional geometry, the relationship can be expressed in various ways, including:

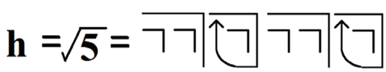
$$\frac{1}{\Phi} = \frac{\sqrt{5} - 1}{2}$$

Another is: $\sqrt{5} = 1 + 2\frac{1}{\Phi}$

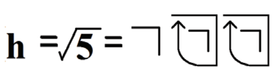
Now let's apply our diagonal rule to the 1:½ rectangle, where t = ½ = :



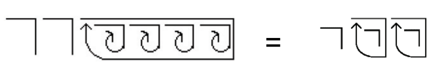
When the two halves in the re-entry space are added this give the diagonal as ,which translates to ½ + $^{1}/\Phi$ as per our diagram above. As this 1:½ rectangle doubles to give 2:1 rectangle, we get √5 by doubling this result:



Add the two halves:



Remember = and so in this result we have the equivalent of , which from Table 11 gives two of $^{1}/\Phi$. Thus, the full expression converts √5 to $1 + 2\ ^{1}/\Phi$.
Next, let's find √5 directly by our diagonal rule, where for the 1:2 rectangle t = 2 =. The result is already found in Table 11. And so we now have two infinite expressions for √5:

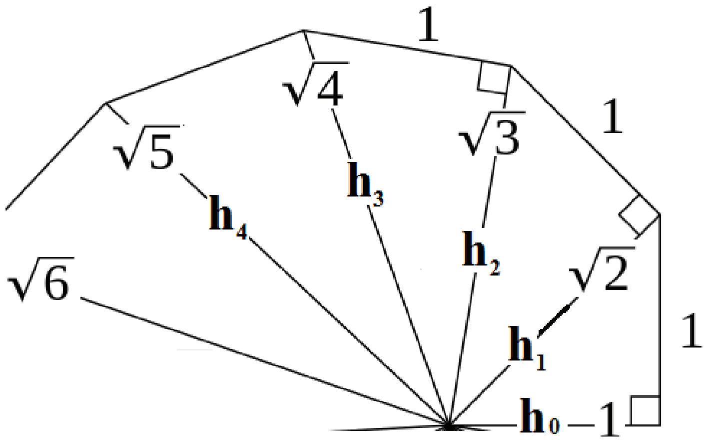


which is a neat way of saying $2 + (\sqrt{5} - 2) = 1 + 2^{1}/\Phi = \sqrt{5}$.

14. The Spiral of Theodorus

Another way to arrive at a measure for √5 is also revealing. This is by using our diagonal rule to build the spiral of Theodorus.[4] This spiral is a geometric expansion generating the square roots of the natural numbers in series through hypotenuse-for-side substitution. Here are the first stages of this progression:

4. The name of this spiral is derived from a passage in Plato's *Theatetus* [p. 147D] that may be referring to it in a discussion of irrational square roots.



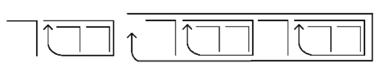
The first right angle triangle in the series has sides 1:1. Pythagoras' theorem gives the hypotenuse length of √2. The next triangle has sides 1:√2 and so an hypotenuse calculated as √3. The next has sides 1:√3. And so forth the series emanates √2, √3, √4…. But note that the unit line from where the series begins is a limit-origin in the Pythagorean sense if it is regarded as a triangle of sides ratio 1:0. Applying Pythagoras' theorem we get an hypotenuse of √1, which is identical in length and position with its unit "side." Thus, in fact the series begins, √1, √2, √3…

In our interpretation, the spiral builds though substituting the previous h for the next t. Thus, $t_{n+1} = h_n$. Starting with the limit-origin with side ratio 1:0, we have $t_0 = 0$. Remember that in laws of form arithmetic the null value is unmarked.

Thus, if **h = t t t** and t is unMarked then: $h_0 =$. This gives the hypotenuse of the limit triangle as √1 = 1 = = .

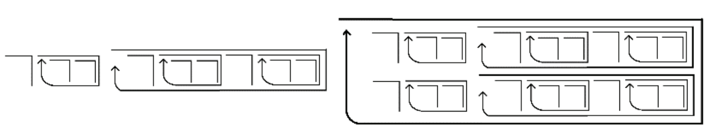
If $h_0 = t_1 =$, then apply the rule again: $h_1 =$, which we know already from Table 11 is the square's diagonal, √2.

If $h_1 = t_2$, again apply the rule:

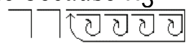
$h_2 =$ 

By Pythagoras' theorem, the hypotenuse of this 1:√2 triangle is √3, which is exactly what this expression generates by fractional analysis. Notice, as we build the spiral, how each new expression contains the previous one. Thus √3 contains three instances of √2, which each contain three instances of √1.

Follow the rule one more time and we get:

$h_3 =$ 

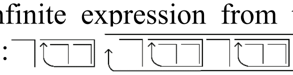
This takes us to a special place in the spiral. If drawn to scale on the page, inspection reveals that h_3 is double h_0, which is 2. The same result is found by Pythagoras' theorem, where the hypotenuse of this $1{:}\sqrt{3}$ triangle is $\sqrt{((\sqrt{3})^2+1^2)} = \sqrt{4} = 2$. And, indeed, fractional analysis of this large 3rd degree expression also converges to $h_3 = 2$.

The finding of this first perfect square (by whatever method) means that the next stage need not build on this larger 3rd degree expression to arrive at h_4 or $\sqrt{5}$. Instead we can start again in the finite because $h_3 = t_4 = \sqrt{4} = 2 =$ ⏋⏋. Apply the rule again and h_4 presents equivalent to  for $\sqrt{5}$ as given in Table 11.

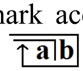
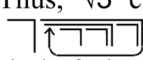
Continue on up the spiral and the expressions again grow by degree until $\sqrt{9}$, the next perfect square. This means that the value of t for $\sqrt{10}$ returns to a finite number, which is triple h_0, so $t = 3 =$ ⏋⏋⏋.

Notice how the degree of the minimum expression of h is one more than the minimum degree expression of t. Continue up the spiral and h ascends into higher and higher degrees as the distance from the previous perfect square increases. This process provides an orderly way of finding infinite expressions for square roots, but they can get rather large! Thankfully, there is another way to find whole number square roots that gives simpler expressions remaining in the 1st degree.

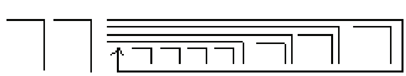
15. Infinite Expressions for Square Roots Derived From Their Continued Fractions

Consider again $\sqrt{3}$. This is not only the diagonal of the $1{:}\sqrt{2}$ square, but also the diagonal of the cube. We already have an infinite expression from the spiral of Theodorus that can be reduced to the 2nd degree: .

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}$$

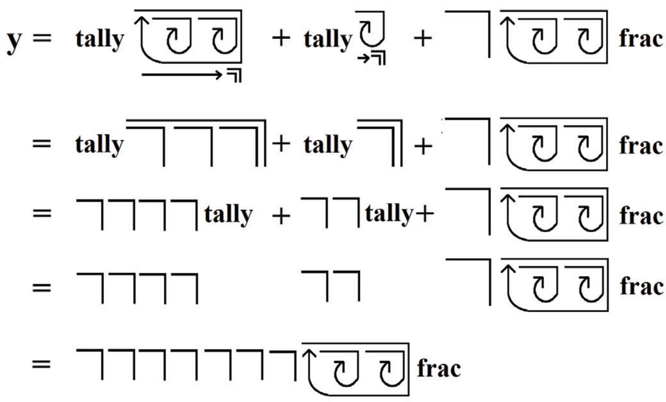
But the standard continued fraction for $\sqrt{3}$ looks even simpler. Notice the $1 \leftrightarrow 2$ alternation in the denominator. Such an alternation can be expressed in our notation by punctuating the divisions with an inserted mark according to the *Laws of Form* "E1" archetype , as discussed in section 2 above. Thus, $\sqrt{3}$ can be expressed simply in 1st degree: .

Next, consider $\sqrt{7}$. By the spiral of Theodorus, we get an expression in 3rd degree similar to the large expression for $\sqrt{4}$ above. Its continued fraction is also relatively complex with a 4-phase cycle of denominators: 1, 1, 1, 4. However, when translated into our arithmetic, it remains in 1st degree of infinity, if only with four stages:



All square roots of natural numbers have eventually periodic continued fraction and so they can all be translated into a 1st degree re-entry expressions using this archetype.

Such infinite rational expressions for conventional irrational numbers can easily be incorporated into equations and then merged at the level of tally abstraction. For example here is $y = 2^2 + 2 + \sqrt{2}$:



Notice how by the last two lines the abstracted tallies, 4 and 2, concatenate in the Fractional interpretation to give one expression with both finite and infinite elements.

16. Further Developments

Thus ends our introduction to the geometry. There is much more to say but that can wait for another time. Meanwhile, others can make their own explorations. To aid such work, a simple computer program has been developed and made freely available on GitHub (Hughson, 2018). This program, "Dynamics of Form," facilitates the analysis of finite and infinite expressions in all the ways discussed above and more. In the first place, the program could be used to confirm the pattern of generation and the fractional analysis of some of the larger expressions provided above. It could also be used to advance exploration of this geometric application in the 3rd and 4th degrees.

The dynamics of form program could also be used to explore other interpretations. Those interpretations of immediate appeal involve convergent series. Consider that we might go back and pay attention to the alternation of Brownian value states down the periods of generation. This alternation could be interpreted as an oscillation positive (marked) and negative (unmarked) around a neutral centre. Thus, for example, the gnomon series for the triangular numbers would be 1, -2, +3, -4, +5… If each gnomon is then interpreted in ratio with the origin, then the triangular number series generates $+{}^1/_1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5 \ldots$, which is the alternating harmonic series, which sums to the natural logarithm of 2. On this same interpretation, the square number series generates ${}^1/_1 - {}^1/_3 + {}^1/_5 - {}^1/_7 + {}^1/_9 \ldots$, which sums to Leibniz's expression for ${}^\pi/_4$. Thus, such an interpretation could be used to build, from our square-cubic geometry, a geometry of circle, cones and spheres, and also a geometry of natural growth and decay.

For now, let's finish by considering the emergent structure of this Boolean mathematics. In doing so, there is no suggestion that our version of Boolean mathematics—with its peculiar beginning in the higher degrees of laws of form—is the exclusive approach to geometry in the form. Rather, this structure is only outlined here so that it may be compared with our received conventional structure of mathematical reasoning in a critical evaluation of both.

17. Classification of Numbers in Boolean Mathematics

All Boolean mathematics begins with the principle-origin. This arithmetic unity (containing the null as its negation) is the form of distinction. It is topologically expressed by a circle on the plain page, and it is notated thus: $\overline{\quad}|$.

Our Boolean mathematics finds its elementary arithmetic progression in the elementary progression of the form re-entering its own inner space, notated thus: $\overline{\curvearrowleft\,}|$. The periods of this re-entry are the elementary ordinal number series. Each number in this series is considered a delimitation of the infinite progression of the re-entry. Thus, the number 3, expressed as $\overline{\overline{\overline{\quad}|}|}|$, is the delimitation of the progression of $\overline{\curvearrowleft\,}|$ to the 3rd period. Other numbers are generated by complexities of re-entries within re-entries and the delimitation of such higher degree emanations.

Logistical counting and calculation comes though interpretations of the ordinal arithmetic. For example, cardinal number is derived from ordinal numbers through tallying Marks in the ordinal expression. For example:

$$\textbf{tally}\,\overline{\overline{\quad}|}| = \overline{\quad}|\,\overline{\quad}|\,\textbf{tally}$$

Pre-distinguished things are counted in this way through a one-to-one correspondence with successive members of the ordinal progression, and then through abstraction of the tally to give a cardinal number.

In a similar way space is metered. For example, length can be measured by counting paces in the periods of the elementary ordinal generator, $\overline{\curvearrowleft\,}|$, and then abstracting the tally. Square space can be measured in the same way through the square number generator, $\overline{\curvearrowleft\,\overline{\curvearrowleft\,}|\,\overline{\curvearrowleft\,}|}|$. For example, the first in the square series is measured to have an area of 4 thus:

$$\text{tally}\;\overline{\overline{\quad}|\,\overline{\quad}|\,\overline{\quad}|}| = \overline{\quad}|\,\overline{\quad}|\,\overline{\quad}|\,\overline{\quad}|\;\text{tally.}$$

Similarly, cubic space can be measured by abstraction using the cubic generator.

Fractions and square roots are generated through another interpretation, where $\overline{X}| = \frac{1}{X}$. The square roots are not irrational, but *infinite ratios*. The fractional interpretation can be merged with interpretation by tally abstraction. This allows arithmetic expressions with finite and infinite components.

Numbers may be classified according to this general structure. The first class is the form of the first distinction, $\overline{\quad}|$. The second class constitutes the members of the elementary ordinal series generated by elementary re-entry, . After that comes infinite expressions involving re-entries. These can be classed in a hierarchy according to degrees of re-entry.

Finite numbers are then classified according to how they are generated. For example, consider the 2nd member in the 2nd degree square series, , which is . This could be classed as a finite 2nd degree number, or, geometrically speaking, as a plane number. Likewise, finite numbers generated by delimitations of 3rd degree generation would be finite solid numbers, and so forth, proceeding by analogy with the Pythagorean hierarchy of dimensional magnitudes into higher degrees.

A completely separate class of numbers comes though interpretations by tally abstraction and fractional analysis. Tallies are simple finite zero order numbers providing a cardinal count. Fractions consist of finite ratios of various orders, as well as infinite ratios of various degrees.

This classification of numbers and ratios by degree offers an ordinal hierarchy of infinity as an alternative to the hierarchy based on Cantor's distinction between countable and uncountable transfinite cardinals (Cantor, 1891). Consider for example our measurement of rectangles with rational side lengths. If only by translation from their continued fractions, their diagonals are all measurable without venturing beyond the 1st degree of the arithmetic (and we can see this in their very notational expression). For rectangles in this class, we might first imagine that the ratio of their sides (t) all lie on a continuum of finite measures, expressible in the finite arithmetic. We might then imagine that their diagonals lie on a continuum in the 1st degree of infinity.

These infinite numbers are also found to have inherent shape. When our Boolean arithmetic is applied to geometry, conventional equations of plane and linear numbers do not hold. For example, $2^2 \neq 4_{\text{linear}} \neq 2 \times 2$. This is because each expression has a different shape due to its generation. They are not equal but they are similar by tally equality, and it is only through tally abstraction that numbers generated in different ways can be reduced to conventional arithmetic values and subject to conventional operations like addition and multiplication. It is also only in this way that discrete whole number values concatenate with fractions and square roots. Otherwise, the notation of laws of form retains the shape of these geometric numbers. Whether this retention can help build a richer, more consistent mathematics remains to be seen.

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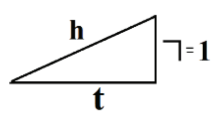
Appendix

The Diagonal Rule and Pythagoras' Theorem

This appendix shows a way that Louis Kauffman has found to derive the diagonal rule from Pythagoras' theorem.

The diagonal rule:

$$\mathbf{h} = \mathbf{t}\,\overline{\mathbf{t\,t}|} \qquad \text{where}$$



Pythagoras' theorem applies:

$$h^2 = t^2 + 1^2$$

$$h = \sqrt{(t^2 + 1)}$$

Firstly, the rule must be translated from our fractional analysis into conventional algebra.

This translation follows the fractional interpretation with its rule: $\overline{X|} = \frac{1}{X}$.
The diagonal rule has a finite component **t,** and an infinite component $\overline{\mathbf{t\,t}|}$.
It is the infinite component that requires the most attention.

The infinite component can be expressed as:

This translates to this infinite continued fraction:

$$\cfrac{\mathbf{1}}{\mathbf{t+t} + \cfrac{\mathbf{1}}{\mathbf{t+t} + \cfrac{\mathbf{1}}{\mathbf{t+t} + \cfrac{\mathbf{1}}{\dots}}}}$$

This fraction can be expressed as a recursive equation:

$$\gamma = \frac{1}{2t + \gamma}$$

Now add the finite component to get the diagonal rule expressed in conventional algebra:

$h = t + \gamma$ where $\gamma = \frac{1}{2t+\gamma}$. Call this DR.

The next step is to convert the recursive component γ into a quadratic equation and solve for γ:

$$\gamma = \frac{1}{2t + \gamma}$$

$$\gamma\,(2t + \gamma) = 1$$

$$\gamma\,(2t + \gamma) - 1 = 0$$

$$\gamma^2 + 2t\gamma - 1 = 0$$

Using the quadratic formula, the recursive component can be expressed as:

$$\gamma = -t + \sqrt{(t^2 + 1)}$$

Next, add **t** to both sides of this equation:

$$t + \gamma = \sqrt{(t^2 + 1)}$$

This gives DR equal to Pythagoras' theorem:

$$h = t + \gamma = \sqrt{(t^2 + 1)}$$

Thus, we have found that our diagonal rule is derived from Pythagoras' theorem.